

Analytical approach to the $D3$ -brane gravity dual for $3d$ Yang-Mills theory

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Abstract

The complexity of “top-down” string-dual candidates for strongly-coupled Yang-Mills theories and in particular for QCD almost always prohibits their exact analytical or even comprehensive numerical treatment. This impedes both a thorough quantitative analysis and the development of more realistic gravity duals. To mitigate these impediments, we devise an analytical approach to top-down duals on the basis of controlled, uniformly converging high-accuracy approximations for the normalizable string modes corresponding to gauge-theory states. We demonstrate the potential of this approach in Witten’s dual for $3d$ Yang-Mills theory, i.e. in the near-horizon limit of non-extremal $D3$ -branes, compactified on S^1 . We obtain accurate analytical approximations to the bulk modes which satisfy the boundary conditions exactly. On their basis, analytical results for masses, sizes, pole residues and correlation functions of glueball excitations are derived by spectral methods. These approximations can be systematically improved and rather universally adapted to more complex gravity duals.

I. INTRODUCTION

Since their inception nearly two decades ago, gauge/string dualities keep generating profound insights into strongly-coupled gauge theories and have led to new perspectives and groundbreaking results over a broad range of physics subfields [1].

Among the pioneering and most intensely studied application areas for these holographic dualities is QCD, the theory of the strong interactions, in its infrared regime. Hence, various increasingly sophisticated and realistic string-theory-based (i.e. “top-down”) dual candidates for strongly-coupled large- N Yang-Mills theories and QCD were constructed. In order to be at least halfway accessible by contemporary solution methods, most of them use the gravity approximation for the string dynamics [16]. This requires weak bulk space-time curvatures and entails several well-understood limitations [1] e.g. in the description of asymptotic freedom or linear Regge trajectories.

Even within these restrictions, however, the technical (and sometimes conceptual) complexities encountered when attempting to calculate gauge-theory quantities beyond approximate mass spectra can be quite formidable. Part of these problems arise from the absence of analytical solutions for even the simplest bulk quantities in practically all top-down duals. Moreover, numerical approaches provide no fully satisfactory alternative. The numerical evaluation of more complex amplitudes like correlators is e.g. typically hampered by renormalization issues and often requires specifically designed, non-universal methods. In addition, numerical results reveal less of the underlying mathematical structure whose patterns and systematics often provide useful insights and intuitive guidance.

To improve this situation, we intend to develop and adapt more sophisticated analytical approximation methods to deal with top-down gravity duals. They should be controlled (i.e. systematically improveable) and as transparent and universally applicable as possible. In particular, they should provide accurate analytical expressions for a complete set of dual quantities from which analytical results for all desired gauge-theory observables can be derived. In this letter we propose the whole tower of (normalizable) dual string modes, corresponding to the “hadrons” of interest, as a promising such set.

In the following, we demonstrate the feasibility of our approach in Witten’s top-down gravity dual for three-dimensional Yang-Mills theory (YM_3) [2–4] which emerges from N parallel, non-extremal $D3$ branes of type IIB string theory in the near-horizon limit, with

one of its flat dimensions compactified on a circle. This dual model suggests itself for our purposes because it is technically simple and reproduces important properties of the gluonic sector of QCD (including confinement).

II. GRAVITY DUAL FOR 3d YANG-MILLS THEORY

In Ref. [2], Witten proposed a string dual for $SU(N)$ Yang-Mills theory in $d = 2 + 1$ dimensions at large N and for large 't Hooft coupling $\lambda = g_{YM}^2 N$. This dual is based on the AdS/CFT correspondence between $\mathcal{N} = 4$ superconformal Yang-Mills theory in $d = 3 + 1$ dimensions and a stack of N coincident $D3$ -brane solutions of type IIB string theory in 10 dimensions (with R-R charge N), or for small spacetime curvature by IIB supergravity [?]. Conformal symmetry and supersymmetry are broken by generalizing the correspondence to non-extremal $D3$ black-brane solutions with Hawking temperature T_H .

One then decouples the near-horizon or “throat” region from the 10d Minkowski gravity (staying outside the horizon) and analytically continues to Euclidean time $\tau = -it$. This requires to periodically identify $\tau + \beta_H \simeq \tau$ on the circle $S^1(R_\tau)$ and to remove a cone singularity in the (τ, r) subspace by fixing the Hawking temperature to $T_H = (\pi R)^{-1}$ with the horizon at $r_0 \simeq \pi T_H R^2 = R$. In the Fefferman-Graham coordinate $z = R^2/r$, the metric then becomes

$$ds^2 = g_{MN} dx^M dx^N = \frac{R^2}{z^2} \left[\left(1 - \frac{z^4}{R^4} \right) d\tau^2 + \sum_{i=1,2,3} dx_i^2 + \left(1 - \frac{z^4}{R^4} \right)^{-1} dz^2 \right] + R^2 d\Omega_5^2 \quad (1)$$

where x_3 is reinterpreted as the Euclidean time. (The associated dilaton is constant.)

On the gauge-theory side the above procedure corresponds to compactifying the Euclidean $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills theory (i.e. the low-energy theory of the N coinciding $D3$ branes) on $\mathbb{R}^3 \times S_\tau^1$, with anti-periodic, supersymmetry-breaking boundary conditions for the fermions around S_τ^1 . Hence the adjoint fermions acquire tree-level masses $\sim 1/R_\tau$ (thus breaking conformal symmetry) and render the scalar matter massive at the one-loop level. For sufficiently small R_τ the adjoint matter decouples and only the gauge fields on \mathbb{R}^3 remain massless (protected by the gauge symmetry), with their dynamics approaching YM_3 (with an UV cutoff $\sim 1/R_\tau$).

The background geometry (1) is dual to the gauge-theory vacuum. Its scalar excitations,

the 0^{++} “glueballs”, are associated with the (gauge-invariant) operator

$$\mathcal{O}_{0^{++}}(x) = \text{tr} \{F_{\mu\nu} F^{\mu\nu}\} \quad (2)$$

which in turn is dual to a bulk field with identical quantum numbers. The corresponding deformation $S \rightarrow S + \int d^d x \varphi^{(s)}(x) \mathcal{O}_{0^{++}}(x)$ of the gauge-theory action changes the Yang-Mills coupling $g_{YM}^2 = g_s = \exp \varphi^{(s)}$ and thus the (boundary) value $\varphi^{(s)}$ of the dilaton field φ . This indicates that φ is the bulk field dual to $\mathcal{O}_{0^{++}}$.

The GKPW relation [5] then identifies the generating functional of the glueball correlation functions as

$$Z[\varphi^{(s)}] = \left\langle e^{\int d^d x \varphi^{(s)}(x) \mathcal{O}_{0^{++}}(x)} \right\rangle \xrightarrow{N \gg \lambda \gg 1} e^{-S^{(onsh)}[\varphi^{(s)}(x) = \varphi(z=0, x)]} \quad (3)$$

with the string partition function in the classical limit. The latter is determined by the on-shell action $S^{(onsh)}$, i.e. by the classical dilaton action (after dimensional reduction on S^5 , i.e. for φ restricted to $\text{SO}(6)$ singlets)

$$S_{0,m}[\varphi] = \frac{1}{2\kappa^2} \int d^5 x \sqrt{|g|} \left[g^{MN} \partial_M \varphi \partial_N \varphi + m_5^2 \varphi^2 + O(\alpha'^n) \right] \quad (4)$$

($\kappa^2/2$ is the five-dimensional Newton constant) in the dual background geometry (1) evaluated at the extrema of Eq. (4). The latter are the solutions of the Laplace-Beltrami equation

$$(g^{\tau\tau} \partial_\tau^2 + \Delta_z + g^{ij} \partial_i \partial_j - m_5^2) \varphi(x, \tau, z) = \int d^3 k e^{iqx} (\Delta_z - a^{-2} q^2 - m_5^2) \varphi(q; z) = 0 \quad (5)$$

after dimensional reduction on $S^1(R_\tau)$, i.e. for τ -independent φ . Here $\varphi(q; z)$ is the $3d$ Fourier transform of the flat boundary coordinates of the dilaton field and

$$\Delta_z \equiv \frac{1}{\sqrt{g}} \partial_z \sqrt{g} g^{zz} \partial_z = \frac{z^5}{R^5} \partial_z \left(\frac{R^3}{z^3} - \frac{z}{R} \right) \partial_z \quad (6)$$

is the radial Laplacian. A Frobenius analysis shows that the normalizable (non-normalizable) solutions of Eq. (5) behave as z^d ($z^{\Delta-d}$) near the boundary. For the (classical) conformal dimension $\Delta = 4$ of the interpolator (2) the AdS/CFT dictionary implies $m_5^2 R^2 = \Delta(\Delta - d) \stackrel{\Delta=d=4}{=} 0$, as expected for massless dilaton modes.

As motivated in the introduction, we intend to construct approximate solutions of the radial dilaton equation

$$[a^2(z) \Delta_z - q^2] \varphi(q; z) = 0 \quad (7)$$

which are normalizable and regular at the horizon. Normalizability selects the subleading solutions which obey the UV boundary condition $\varphi(q; z) \xrightarrow{z \rightarrow 0} z^4$. These solutions have discrete eigenvalues $q_n^2 = -m_n^2$ for $n = 1, 2, \dots$ where m_n is the mass of the n -th radial glueball excitation [2]. More explicitly, the n -th eigensolution $\psi_n(z) := \varphi(q_n; z)$ solves the radial equation

$$[\partial_z^2 + t_1(z) \partial_z + m_n^2 t_2(z)] \psi_n(z) = 0 \quad (8)$$

(a 2nd-order Fuchsian differential equation) whose coefficient functions

$$t_1(z) = -\frac{z^4 + 3R^4}{z(R^4 - z^4)}, \quad t_2(z) = \frac{R^4}{R^4 - z^4} \quad (9)$$

have four regular singular points at $z = 0, \pm R, \infty$. Hence no general, analytical solutions for Eq. (8) are known.

III. UNIFORM ANALYTICAL APPROXIMATIONS TO THE STRING MODES

In this section, we will derive controlled, uniformly convergent analytical approximations to all eigenmodes ψ_n of Eq. (8) under the stated boundary conditions. Since we are dealing with a Sturm-Liouville problem which is not of the standard Schrödinger type (see below), we will rely on a generalization of the WKB method (GWKB) [17] and just sketch the main steps here, relegating more details to a subsequent, more comprehensive publication [10].

To impose the boundary conditions at the singular points $z = 0, R$, it is convenient to transform to the new radial coordinate

$$y(z) = \ln \left(\frac{R^2}{z^2} - 1 \right) \quad (10)$$

which maps the horizon to $y = -\infty$ and the AdS boundary to $y \rightarrow +\infty$. Equation (8) is thereby converted into the quasi-Schrödinger form

$$\left[-\partial_y^2 + \frac{1}{\epsilon^2} V_1(y) + V_2(y) \right] u(y) = 0. \quad (11)$$

with the potentials

$$V_1(y) = -\frac{e^y}{(1 + e^y)(2 + e^y)} \quad (12)$$

and

$$V_2(y) = \frac{1}{2} \frac{(3 + 4e^y) e^y}{(1 + e^y)(2 + e^y)} - \frac{1}{4} \frac{(3 + 2e^y)^2 e^{2y}}{(1 + e^y)^2 (2 + e^y)^2} \quad (13)$$

where the eigenvalues appear in the combination $\epsilon^{-2} = (m^2 R^2) / 4$.

A global and uniform (asymptotic) expansion of the solutions of Eq. (11) can be organized in the familiar form

$$u(y) \sim \exp \left(\frac{1}{\epsilon} \sum_{k=0}^{\infty} S_k(y) \epsilon^k \right), \quad \epsilon \rightarrow 0+ \quad (14)$$

whose ϵ dependence implements a standard version of “distinguished balance” [11]. The leading-order solution is thus of $O(\epsilon^0)$. Equation (11) implies that the potentials $V_{1,2}$ contribute at different orders to the asymptotic expansions. This requires a non-standard two-scale power counting analysis and the corresponding GWKB matching procedure around the turning points of Eqs. (12) and (13).

For approximations that are sufficiently accurate to satisfy both the regularity ($u(-\infty) = c$) and normalizability ($u(+\infty) = \lim_{y \rightarrow \infty} c e^{-y}$) boundary conditions exactly, one has to push the expansion (14) to $O(\epsilon^2)$, i.e. to next to next to leading order (N²LO). A uniform approximation further requires to match $u(y)$ near both boundaries to exact solutions of approximate radial field equations, expanded to the appropriate orders of e^y and ϵ . (This becomes necessary since to N²LO the ansatz (14) diverges at both boundaries $y \rightarrow \pm\infty$.) The resulting two matching relations (or, equivalently, the two boundary conditions) can be simultaneously satisfied only for the discrete spectrum

$$\epsilon_n = \frac{2}{m_n R} = \frac{{}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, 1\right)}{\left(n + \frac{1}{2}\right) \pi} \xrightarrow{n \gg 1} \frac{2^{-1/2} \sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2 \sqrt{n(n+1)}}, \quad n = 1, 2, \dots \quad (15)$$

of eigenvalues (where ${}_2F_1$ is Gauss’ hypergeometric function [12]). Since Eq. (15) does not receive N²LO corrections, its large- n version can alternatively be obtained from an NLO Bohr-Sommerfeld type integral between the two turning points of V [6].

After transforming the $u_n^{(N^2LO)}(y)$ eigenmodes back to the original z coordinate via

$$\psi_n(z) = \frac{z^2}{R^3} \sqrt{\frac{1}{R^2 + z^2}} u_n(y(z)) \quad (16)$$

one ends up with uniform, global N²LO solutions to the radial field equation (8) which consist of three matching parts. The first one is the near-horizon solution

$$\psi_{R,n}(z) \xrightarrow{z \rightarrow R} c_n \frac{z^2}{R^3} \frac{e^{-\frac{1}{2} \sqrt{\frac{3}{\epsilon_n^2} - \frac{11}{2^2}} \frac{R^2 - z^2}{z^2}}}{(R^2 + z^2)^{1/2}} {}_1F_1 \left(\frac{1}{2} - \frac{1}{4} \frac{\frac{2}{\epsilon_n^2} - 3}{\sqrt{\frac{3}{\epsilon_n^2} - \frac{11}{2^2}}}, 1; \sqrt{\frac{3}{\epsilon_n^2} - \frac{11}{2^2}} \frac{R^2 - z^2}{z^2} \right) \quad (17)$$

(where ${}_1F_1$ is Kummer's confluent hypergeometric function [12]). In the intermediate region, the solution is

$$\begin{aligned} \psi_{int,n}(z) \stackrel{R \ll z \ll 0}{\simeq} & c_n \sqrt{\frac{\epsilon_n}{\pi}} \frac{z^{3/2}}{R^{5/2} (R^4 - z^4)^{1/4}} \times \\ & \times \cos \left\{ -\frac{2}{\epsilon_n} \left[\frac{z}{R} {}_2F_1 \left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \left(\frac{z}{R} \right)^4 \right) - \frac{\pi^{3/2}}{2^{3/2} \Gamma(\frac{3}{4})^2} \right] - \frac{\pi}{4} \right. \\ & \left. - \frac{\epsilon_n}{16} \left[\frac{15 \frac{R}{z} - 13 \left(\frac{z}{R} \right)^3}{\sqrt{1 - \left(\frac{z}{R} \right)^4}} + 4 \left(\frac{z}{R} \right)^3 {}_2F_1 \left(\frac{3}{4}, \frac{1}{2}, \frac{7}{4}, \left(\frac{z}{R} \right)^4 \right) - 4 {}_2F_1 \left(\frac{3}{4}, \frac{1}{2}, \frac{7}{4}, 1 \right) \right] \right\}. \end{aligned} \quad (18)$$

This is the only non-monotonic segment (for $n > 1$) which therefore contains the oscillatory part of the complete solution. Finally, near the boundary the solution becomes

$$\begin{aligned} \psi_{0,n}(z) \stackrel{z \rightarrow 0}{\simeq} & (-1)^{n-1} \frac{c_n}{2} \left(\frac{1}{\epsilon_n^2} + \frac{3}{2} \right) \frac{z^4}{R^3 (R^2 + z^2)^{1/2} (R^2 - z^2)} \\ & \times {}_1F_1 \left(\frac{3}{2} - \frac{\frac{1}{\epsilon_n^2} + \frac{3}{2}}{2 \sqrt{\frac{3}{\epsilon_n^2} + \frac{9}{4}}}, 3, 2 \sqrt{\frac{3}{\epsilon_n^2} + \frac{9}{4}} \frac{z^2}{R^2 - z^2} \right). \end{aligned} \quad (19)$$

The above solutions have the correct number of $n - 1$ nodes, as expected on general grounds and confirmed by the numerical solutions. (In contrast, the NLO solutions have one additional, spurious node). As required by normalizability, the modes exhibit sub-leading behavior near the boundary, i.e.

$$\psi_n(z) \stackrel{z \rightarrow 0}{\longrightarrow} c_n (-1)^{n-1} \frac{1}{2} \left(\frac{1}{\epsilon_n^2} + \frac{3}{2} \right) \frac{z^4}{R^6}, \quad (20)$$

and are regular at the horizon where they attain the finite values

$$\psi_n(R) = \frac{c_n}{\sqrt{2} R^2}. \quad (21)$$

(Ortho-) normalizing the modes with respect to their Sturm-Liouville inner product

$$\int_0^R dz \sqrt{g} g^{zz} \psi_m(z) \psi_n(z) = \int_0^R dz \frac{R^3}{z^3} \psi_m(z) \psi_n(z) = \delta_{mn} \quad (22)$$

fixes the overall coefficients c_n . The N²LO approximations to these integrals can be derived from the $u_n(y)$ by exploiting the relation

$$\int_0^R dz \frac{R^3}{z^3} [\psi_n(z)]^2 = -\frac{1}{2R^3} \int_{-\infty}^{+\infty} dy V_1(y) [u_n(y)]^2 \quad (23)$$

n	1	2	3	4	5	6	7	8	9	10
$\psi_n^{(N^2LO)^2}(R)$	4.30662	7.17770	10.0488	12.9199	15.7909	18.6620	21.5331	24.4042	27.2753	30.1463
$\psi_n^{(num)^2}(R)$	4.28354	7.17546	10.0481	12.9196	15.7908	18.6619	21.5330	24.4041	27.2752	30.1463

TABLE I: Comparison of the N²LO approximation (25) and the numerical results for the value $\psi_n(R)$ of the n -th eigenmode at the horizon (for $R = 1$).

which simplifies the integration over the special functions and the consistent two-scale power counting. The final result to N²LO accuracy,

$$c_n^2 \simeq \frac{\pi^2 R^3}{{}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, 1\right) K(-1)} \left(n + \frac{1}{2}\right) = 4\sqrt{2} \frac{\Gamma\left(\frac{3}{4}\right)^3}{\Gamma\left(\frac{1}{4}\right)} R^3 (2n + 1) \quad (24)$$

(where $K(n)$ is the complete elliptic integral of the first kind [12]), is surprisingly simple and accurate (see below).

By construction, the N²LO solutions converge uniformly to the exact solutions for small ϵ , i.e. for large n . Their accuracy turns out to be unexpectedly high already for the smallest n , however. While the nodeless ground-state ($n = 1$) mode is still the least well approximated [18], the accuracy grows already to the sub-percent level for $n = 2$, as revealed by comparison with the numerical solution in Fig. 1. From $n \gtrsim 4$ the N²LO and numerical solutions are identical within plot resolution (cf. Fig. 2). Another rather stringent test for the mode accuracy (and normalization) provides the comparison of the N²LO horizon values

$$\psi_n^2(R) = 2\sqrt{2} \frac{\Gamma\left(\frac{3}{4}\right)^3}{\Gamma\left(\frac{1}{4}\right)} \frac{1}{R} (2n + 1) \quad (25)$$

with their numerical counterparts given in Tab. I. The deviations start from about half a percent for $n = 1$ and decrease rapidly with increasing n .

With the accurate string-mode approximations (17) - (19) we have achieved our first and central objective. On their basis, analytical glueball properties and amplitudes will be derived in the following section.

IV. ANALYTICAL GLUEBALL PROPERTIES

The gauge-theory observables which are most directly related to the normalizable eigenmodes are the glueball masses encoded in their eigenvalues. From Eq. (15) one finds, to

N²LO accuracy, the mass spectrum

$$m_n^2 = \left[\frac{\pi \left(n + \frac{1}{2}\right)}{{}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, 1\right) R}\right]^2 \simeq \frac{1}{2} n(n+1) m_1^2, \quad n \geq 1 \quad (26)$$

which exhibits a finite gap [19] [6]

$$m_1^2 = \frac{2^4}{\pi} \Gamma\left(\frac{3}{4}\right)^4 \frac{1}{R^2} \simeq 11.4843 \frac{1}{R^2} \quad (27)$$

typical for color confinement [20]. A fit to the first 50 numerical eigenvalues corroborates the n dependence of the mass spectrum (26) and yields an almost identical value $m_1^{(fit)2} = 11.4844/R^2$ for the mass gap. The quality of the N²LO approximation is surprisingly good again even for $n = 1$, and excellent from about $n \gtrsim 5$ (cf. Table II).

A second fundamental set of glueball observables are the pole residues or “decay constants” f_n which parametrize the matrix elements

$$\langle 0 | O_{0^{++}}(x) | 0_n^{++}(k) \rangle = \langle 0 | O_{0^{++}}(0) | 0_n^{++}(k) \rangle e^{-ikx} = f_n m_n^2 e^{-ikx}. \quad (28)$$

In contrast to the “local” nature of the masses, the f_n are of “global” origin in the sense that they depend on the overall normalization of the modes,

$$f_n = \frac{1}{\kappa m_n^2} \lim_{\varepsilon \rightarrow 0} \sqrt{g}(\varepsilon) g^{zz}(\varepsilon) \psi'_n(\varepsilon) = \frac{R^3}{\kappa m_n^2} \lim_{\varepsilon \rightarrow 0} \frac{\psi'_n(\varepsilon)}{\varepsilon^3}, \quad (29)$$

(the prime indicates a radial derivative). Evaluating this formula with the behavior (20) and normalization constants (24) of the near-boundary modes (19) shows that the N²LO results can be cast into the form

$$f_n^2 = \frac{1}{3} (2n+1) f_1^2 \quad (30)$$

with

$$f_1^2 = 3\sqrt{2} \frac{\Gamma\left(\frac{3}{4}\right)^3}{\Gamma\left(\frac{1}{4}\right)} \frac{R}{\kappa^2} \simeq 2.153 \frac{R}{\kappa^2}. \quad (31)$$

Again, these results approximate their numerical counterparts very well (cf. Tab. II). A fit to the first 50 numerical f_n values confirms the overall n dependence (30) and yields $f_1^{(fit)2} \simeq 2.148 R/\kappa^2$ [21]. The analytical form of the results reveals in addition a new relation

$$f_n^2 = \frac{c_n^2}{2R^2\kappa^2} = \frac{R^2}{2\kappa^2} \psi_n^2(R) \quad (32)$$

n	1	2	3	4	5	6	7	8	9	10
$m_n^{(N^2LO)2}$	11.48432	34.45296	68.90592	114.84320	172.26480	241.17072	321.56096	413.43552	516.79441	631.63761
$m_n^{(num)2}$	11.58766	34.52698	68.97496	114.91044	172.33117	241.23661	321.62655	413.50092	516.85966	631.70276
$f_n^{(N^2LO)2}$	2.15331	3.58885	5.02439	6.45993	7.89547	9.33101	10.7666	12.2021	13.6376	15.0732
$f_1^{(num)2}$	2.14172	3.58746	5.02404	6.45978	7.89540	9.33097	10.7665	12.2021	13.6376	15.0732

TABLE II: Comparison of the numerical and N²LO results for m_n^2 and f_n^2 .

which corroborates that the residues contain information on the $\psi_n(z)$ over their whole domain $0 \leq z \leq R$.

For related reasons, the pole residues also contain information on the size of the associated gauge-theory bound states. The increasing support of the dual modes (17) - (19) near the horizon in the fifth dimension may e.g. indicate that the glueball sizes remain bounded for large excitation number n . A more explicit relation between f_n and the glueball sizes arises from the (properly renormalized) coincidence limit of the gauge-invariant Bethe-Salpeter amplitudes

$$\chi_n(x) = \left\langle 0 \left| \text{tr} \left\{ F_{\mu\nu} \left(-\frac{x}{2} \right) U \left(-\frac{x}{2}, \frac{x}{2} \right) F^{\mu\nu} \left(\frac{x}{2} \right) \right\} \right| 0_n^{++} \right\rangle \quad (33)$$

(where U is the adjoint color parallel transporter). It shows that the $f_n m_n^2 \sim \lim_{|x| \rightarrow 0} \chi_n(x)$ play the role of the n -th glueball's “wave function at the origin” [22].

Finally, we demonstrate how the GWKB eigenmode approximations (17) - (19) can be used to obtain accurate analytical approximations for more complex gauge-theory amplitudes. An illustrative example is provided by the two-point correlation function of the interpolator (2). Its generating functional (3) is determined by the (Gaussian) on-shell action which reduces Eq. (4) to the surface term

$$S_{\partial M}^{(onsh)}[\varphi] = -\frac{1}{2\kappa^2} \int_{\partial M} d^4x [\sqrt{g} g^{zz} \varphi(x, z) \partial_z \varphi(x, z)]_{z \rightarrow 0} \quad (34)$$

where $\varphi(x, z)$ is a solution of the field equation (5) (which causes the bulk action to vanish). The (boundary) Fourier-transformed solutions corresponding to a finite boundary source $\hat{\varphi}^{(s)}(q)$ can be written as

$$\hat{\varphi}(q, z) = \hat{K}(q, z) \hat{\varphi}^{(s)}(q) \quad (35)$$

where $\hat{K}(q, z)$, subject to the UV boundary condition $\hat{K}(q; \varepsilon \rightarrow 0) = 1$, is the bulk-to-boundary propagator [14]. Inserting the solution (35) into Eq. (34) casts the on-shell action

into the form

$$S_{\partial M}^{(onsh)}[\varphi_s] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \hat{\varphi}_s(-q) \Pi(q) \hat{\varphi}_s(q) \quad (36)$$

where the momentum-space correlator is given by

$$\Pi(q) = i \int d^3 x e^{iq(x-y)} \langle T \mathcal{O}_{0++}(x) \mathcal{O}_{0++}(y) \rangle = \frac{1}{\kappa^2} \left[-\sqrt{g} g^{zz} \hat{K}(q, z) \partial_z \hat{K}(-q, z) \right]_{z=\varepsilon \rightarrow 0}. \quad (37)$$

We now expand \hat{K} into the complete set of normalizable bulk eigenmodes ψ_n , which results in the spectral representation

$$\hat{K}(q, z) = \kappa \sum_n \frac{f_n m_n^2}{q^2 + m_n^2} \psi_n(z). \quad (38)$$

(The coefficients are obtained by noting that both $\hat{K}(q, z)$ and the $\psi_n(z)$ solve the radial field equation (7) and that the $\psi_n(z)$ are orthonormal under the inner product (22).) Plugging Eq. (38) into Eq. (37) then provides the (formal) spectral representation

$$\Pi(q) = \sum_n \frac{f_n^2 m_n^4}{q^2 + m_n^2} =: \Pi^{(3)}(q^2) + P^{(3)}(q^2) \quad (39)$$

for the 2-point correlation function. It contains a divergent subtraction polynomial $P^{(3)}$ which can be extracted as

$$P^{(3)}(q^2) = \Pi(q^2) - q^2 \frac{d\Pi(q^2)}{dq^2} + \frac{1}{2} q^4 \frac{d^2 \Pi(q^2)}{d(q^2)^2} \Big|_{q^2=0} = \sum_{n=1}^{\infty} f_n^2 m_n^2 \left(1 - \frac{q^2}{m_n^2} + \frac{q^4}{m_n^4} \right). \quad (40)$$

The remainder, i.e. the thrice subtracted correlator $\Pi^{(3)}$, is then given in terms of the finite spectral sum

$$\Pi^{(3)}(q^2) = - \sum_{n=1}^{\infty} \frac{q^6 f_n^2}{m_n^2 (q^2 + m_n^2)}. \quad (41)$$

Now it is important to observe that the N²LO expressions (26), (27) for the masses and (30), (31) for the pole residues allow Eq. (41) to be summed analytically. The result is

$$\Pi^{(3)}(q^2) = -\frac{2f_1^2}{3m_1^2} q^4 \left[2\gamma_E - 1 + \frac{m_1^2}{2q^2} + \psi \left(\frac{1}{2} + \frac{\sqrt{1 - 8q^2/m_1^2}}{2} \right) + \psi \left(\frac{1}{2} - \frac{\sqrt{1 - 8q^2/m_1^2}}{2} \right) \right] \quad (42)$$

(where $\psi(x)$ is the digamma function [12]). Equation (42) seems to be the first explicit result for a correlation function in the $D3$ -brane dual for YM_3 . It provides a uniformly

accurate approximation to the exact correlator (evaluated numerically) and a generalizable paradigm [10] for obtaining analytical gauge-theory amplitudes from top-down gravity duals [23].

The coefficient in front of Eq. (42) is determined by the GWKB results (27) and (31) for the lightest glueball as

$$\frac{f_1^2}{m_1^2} = \frac{3}{2^4} \frac{R^3}{\kappa^2}. \quad (43)$$

The Fourier-transformed correlator $\Pi(x)$ shows the expected exponential decay $\Pi(x) \xrightarrow{|x| \rightarrow \infty} \sim \exp(-m_1 |x|)$ generated by the mass gap (27). The large $q^2/m_1^2 \gg 1$ expansion

$$\Pi^{(3)}(q^2) = -\frac{1}{2^3} \frac{R^3}{\kappa^2} q^4 \left[2\gamma_E - 1 + \ln 2 + \ln \frac{q^2}{m_1^2} + \frac{1}{3} \frac{m_1^2}{q^2} - \frac{1}{60} \frac{m_1^4}{q^4} - \frac{25771}{2^8 3^2 5 \times 7} \frac{m_1^6}{q^6} + \dots \right] \quad (44)$$

provides a rather good approximation to the correlator (42) already for $q^2 \gtrsim 3m_1^2$ and bears some resemblance with an operator product expansion (OPE) [15] (although the gravity dual remains strongly coupled in the UV). Besides the conformal logarithm (with prefactor $\sim q^4$ as in YM_4) and “condensate-induced” power corrections, the expansion (44) also contains a term $\sim 1/q^2$ which cannot appear in the OPE. A more thorough analysis and discussion of glueball correlators in top-down duals, obtained by the above methods, will be given elsewhere [10].

V. SUMMARY AND CONCLUSIONS

The main purpose of this letter was to outline a controlled approximation scheme for the analytical treatment of string-based gauge/gravity duals. Our approach is based on uniform high-accuracy approximations for normalizable string modes which are dual to gauge-theory states. The approximate mode solutions are obtained from a generalized WKB expansion driven to at least next-to-next to leading order. Analytical results for gauge-theory properties and amplitudes are then calculated using spectral methods.

We have demonstrated the potential of this approach in one of the simplest top-down models for strong-interaction physics, i.e. in the non-extremal, compactified $D3$ -brane string dual for Yang-Mills theory in three dimensions (with gauge group $SU(N)$ at large N and at strong 't Hooft coupling $\lambda = g_{YM}^2 N$). We derived controlled, global approximations for the dilaton modes dual to scalar glueballs and obtained physically transparent analytical

approximations for the associated gauge-theory amplitudes, including the two-point glueball correlation function.

New results and insights from our analytical treatment comprise (i) the systematics of the mode behavior with increasing radial excitation number, (ii) new relations between glueball properties and global aspects of their dual modes, (iii) highly accurate analytical expressions for the pole residues (or “decay constants”), (iv) insights into the size systematics of the scalar glueball excitations and (v) the first expression for the scalar glueball correlator for which we obtain a uniformly accurate analytical approximation. The latter demonstrates the potential of deriving controlled analytical approximations to “hadronic” amplitudes by exact summation of their spectral representations.

We end by noting several general benefits of our approach. The first is the physical transparency of many results which reveals the systematics of contributions from towers of excited states. The latter are efficiently condensed into comprehensive and often instructive analytical expressions. This represents a clear advantage over numerical solutions. A second general benefit is that the underlying set of approximate string-mode solutions contains all glueball-related information needed for the calculation of other glueball amplitudes, including e.g. form factors and scattering amplitudes. A third strength of our method, finally, is the universality with which it can be adapted to rather different and more complex top-down gravity duals, including e.g. those which do not approach AdS spaces at their UV boundary.

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 - [16] or on highly rotationally excited strings
 - [17] We note that next-to-leading order WKB estimates of eigenvalue (i.e. mass) spectra are known

for several gravity duals, including the one we are considering here (cf. e.g. Refs. [4, 6–8]). We propose N²LO approximations to the *eigenmodes*, however, which contain much more information (see below).

- [18] In fact, it stays slightly discontinuous at the matching points. This presents no relevant practical problems, however, since the ground-state mode has the simplest z dependence and can easily be improved (e.g. variationally [8]).
- [19] Note that the $n = 1$ scalar glueball is the lowest-lying state of the complete excitation spectrum [4].
- [20] The quadratic large- n relation $m_n^2 \sim n^2$ is typical for top-down gravity duals (whereas QCD generates linear trajectories).
- [21] In popular bottom-up models this behavior is not generally reproduced: the f_n grow e.g. similarly with n in the “hard-wall” model while they become asymptotically n -independent in the “soft wall” model [13].
- [22] The precise interpretation of the Bethe-Salpeter “density” is not straightforward, however. A proper size definition should therefore be based on form factors describing conserved charge distributions.
- [23] The exact summation over all excitations requires analytical expressions for all poles and residues with their explicit n dependence. A numerical treatment, truncated to a finite set of low-lying modes, is generally not sufficient.

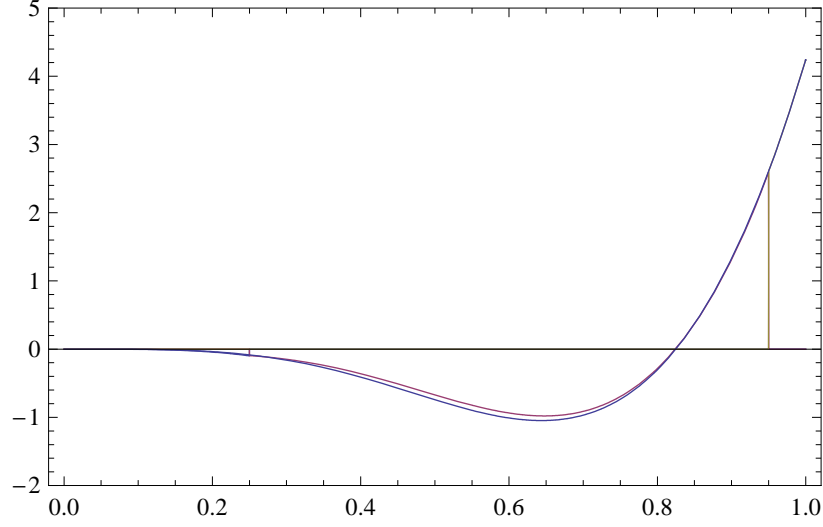


FIG. 1: The (unnormalized) numerical bulk-mode solution (lower curve) and its $N^2\text{LO}$ approximation (upper curve) for $n = 2$. The two vertical line segments indicate the matching points.

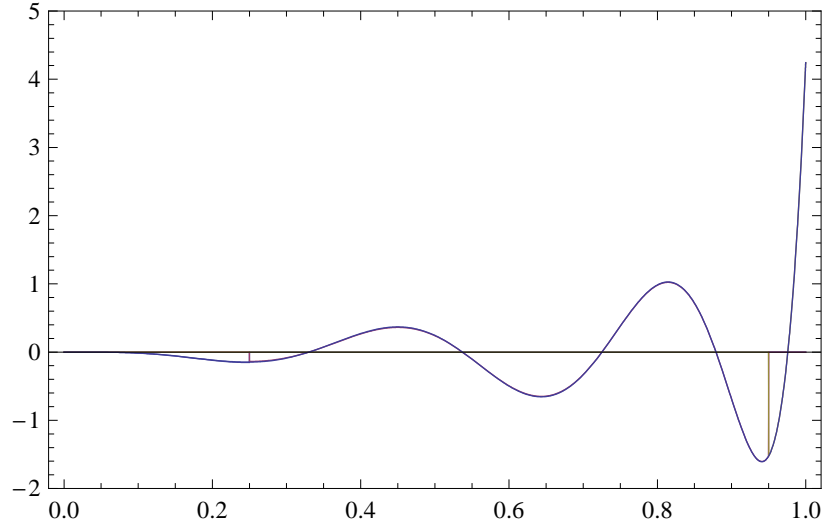


FIG. 2: Same as in Fig. 1, but for $n = 6$. (For $n \gtrsim 4$ the numerical solution and the $N^2\text{LO}$ approximation are indistinguishable within plot resolution.)